Wave component of free-surface Green function

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Summary

The Green function represented by a double Fourier integral associated with the free-surface dispersive and fluid dissipative effects in water wave problems is analyzed. Further to that performed in [6] associated with a real dispersion function, the approximation of the Green function associated with a *complex* dispersion function gives the wave component which keeps the same analytical features as those revealed in [2] but with an exponentially-decreasing amplitude function due to the dissipative effect. The application to steady ship waves with gravity and surface-tension effect as well as fluid dissipation is then performed.

Introduction

With the presence of a free surface, fluid flows underneath are commonly described by an integral representation resultant from the application of Fourier transform technique. In the three-dimensional water wave problems, the double Fourier integral with respect to the horizontal space coordinates and wavenumber vectors are derived to represent flow characteristics such as flow velocity, pressure, velocity potential and free-surface wave profile. In particular, the Green function representing the velocity potential generated by a point source or by a distribution of singularities is of type

$$G = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{\mathcal{N}}{\mathcal{D}} \exp(-i\varphi) \tag{1}$$

where G can be flow velocities, pressure, velocity potential or wave elevation. The term $\mathcal{N}/\mathcal{D}$ is often denoted by $A=\mathcal{N}/\mathcal{D}$ and called as amplitude function so that $\mathcal{N}$ and $\mathcal{D}$ are respectively the numerator and denominator of the amplitude function. The function $\varphi=\alpha x+\beta y$ equal to the product of space vector (x, y) and the vector (α, β) of Fourier variables in the elementary wave function $E=\exp(-i\varphi)$ is called the phase function.

The potential flow represented by (1) associated with $\mathcal{D}=D+i\epsilon\Sigma_1$ with a real function D and a sign function $\Sigma_1(=\pm 1)$ at the limit of $\epsilon \to 0$ has been analyzed in [6]. The decomposition of (1) into wave and local components is obtained. Following the same principle, the most general case of (1) associated with a complex function $\mathcal{D}$ is analyzed here. The case of free-surface flow around an advancing ship is considered. The wave component is obtained by an approximative analysis and expressed by a single integral along the curve(s) defined by $\Re\{\mathcal{D}\} = 0$. The steady ship waves is then presented as an example of applications.

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Green function of free-surface flow

The potential flow due to a ship advancing at a constant speed U along a straight path through a train of regular waves at a frequency of encounter ω is observed from a translating system of coordinates following the mean forward motion of the ship. The x axis is chosen along the path of the ship, z axis vertical and points upward and the mean free surface is taken as the plane $z=0$. The velocity potential is expressed by the sum of a term of simple singularities and a term account of free-surface effect in form of the double Fourier integral (1). The numerator $\mathcal{N}=S\exp(kz)$ with S the spectrum function associated with a distribution of singularity ($S=1$ for a point source) and the complex dispersion function is defined by

$$\mathcal{D} = D + i\mu B \tag{2}$$

in which the real part of the dispersion function is given

$$D = (\alpha-\tau)^2 - k - \sigma^2 k^3 \quad \text{with} \quad k = \sqrt{\alpha^2+\beta^2} \tag{3}$$

and the imaginary part is associated with the small positive parameter $\mu \ll 1$ and the function

$$B = -4(\alpha-\tau)k^2 \tag{4}$$

derived from the dissipative effect in fluid.

In (3), $f = \omega\sqrt{L/g}$ is the nondimensional frequency, $F = U/\sqrt{gL}$ the Froude number and the parameter $\tau = fF$. L and g are the ship length and the acceleration of gravity. Furthermore, the parameter σ in (3) is defined by

$$\sigma = \sqrt{T/(\rho g L^2)}/F^2 \tag{5}$$

representing the ratio between the characteristic wavenumber of capillary waves and that of *gravity* ship waves. In (5), T is the surface tension ($T=0.074$ N/m for the air-water interface at 20°C).

The imaginary part $+i\mu B$ in (2) is associated with the fluid viscosity. The equation system of Stokes or Oseen type is established in [1] and [5]. By assuming that the fluid velocity is composed of a potential part and a rotational part, the free-surface elevation is represented by a double Fourier integral of type (1) with a complex dispersion function similar to (2) where terms of order $O(\mu^{3/2})$ or higher are neglected. The parameter μ for the steady flow is defined in [1] and [5] as

$$\mu = \nu g/U^3 = (1/R)/F^2 \tag{6}$$

with ν the kinematic viscosity and R the Reynolds number.

Wave component of free-surface flow

Since $\mu \ll 1$, the amplitude function $A=\mathcal{N}/\mathcal{D}$ has large variation across the *dispersion* curves defined by $D=0$. It is then natural to make a transform of the double Fourier integral (1) by changing integral variables as :

$$G = \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \, \frac{\mathcal{N}}{\mathcal{D}} \exp(-i\varphi) = \sum_{D=0} \int_{D=0} ds \int_{-N_1}^{N_2} dn \, \frac{\mathcal{N}}{\mathcal{D}} \exp(-i\varphi) \tag{7}$$

since $d\alpha d\beta = dsdn$. The interval of the inner integral is denoted by $(-N_1, N_2)$. In (7), $\Sigma_{D=0}$ means summation over all the dispersion curves, ds is the differential element of arc length of a dispersion curve and dn the differential element of arc length alone curves orthogonal to the dispersion curves.

To evaluate the inner integral, we use the Taylor development at $D=0$ of

$$\mathcal{N} = \mathcal{N}_0 + O(D) \; , \; B = B_0 + O(D) \quad \text{and} \quad \varphi = \varphi_0 + \varphi_0' D + O(D^2) \tag{8}$$

The subscript $(_0)$ indicates values at a dispersion curve $D=0$ and the superscript $(')$ indicates derivatives with respect to D. The derivative of the phase function is evaluated by

$$\varphi' = \frac{\partial \varphi}{\partial D} = \frac{1}{\|\nabla D\|} \frac{\partial \varphi}{\partial n} = \frac{1}{\|\nabla D\|} \nabla\varphi \frac{\nabla D}{\|\nabla D\|} = \frac{\nabla\varphi \nabla D}{\|\nabla D\|^2} \tag{9}$$

where we have used the relations

$$\vec{n} = \nabla D / \|\nabla D\| \quad \text{and} \quad dn = dD / \|\nabla D\| \tag{10}$$

Since $\varphi = \alpha x + \beta y$, we have

$$\varphi' = (x D_\alpha + y D_\beta) / \|\nabla D\|^2 \tag{11}$$

The inner integral in (7) can now be estimated by

$$\int_{-N_1}^{N_2} dn \, \frac{\mathcal{N}}{\mathcal{D}} \exp(-i\varphi) \approx \frac{\mathcal{N}_0}{\|\nabla D\|_0} \exp(-i\varphi_0) \int_{-\infty}^{\infty} dD \frac{\exp(-i\varphi_0' D)}{D + i\mu B_0} \tag{12}$$

The integral on the right hand side is given by

$$\int_{-\infty}^{\infty} dD \frac{\exp(-i\varphi_0' D)}{D + i\mu B_0} = -i2\pi H(B_0 \varphi') \text{sign}(\varphi_0') \exp(-\mu B_0 \varphi_0') \tag{13}$$

according to Lighthill (Eq.54 in [4]). In (13), $H(\cdot)$ is Heaveside's unit function and

$$2H(B_0\varphi')\text{sign}(\varphi_0') = \text{sign}(B_0) + \text{sign}(\varphi_0') \tag{14}$$

so that

$$G \approx -i\pi \sum_{D=0} \int_{D=0} ds \left[\text{sign}(B) + \text{sign}(x D_\alpha + y D_\beta) \right] \frac{\mathcal{E}\mathcal{N}}{\|\nabla D\|} \exp(-i\varphi) \tag{15}$$

in which we have simplified the writing of $(\mathcal{N}_0, B_0, \varphi_0, \|\nabla D\|_0)$ by $(\mathcal{N}, B, \varphi, \|\nabla D\|)$ along the dispersion curves $D=0$, and $\text{sign}(\varphi') = \text{sign}(x D_\alpha + y D_\beta)$ according to

(11). The term $\mathcal{E}$ is definded as $\exp(-\mu B\varphi')$. When $\mu \to 0$, the term $\mathcal{E} \to 1$ since $B < \infty$ along the dispersion curves. At this limit, (15) becomes Eq.27b in [6] which represents the wave component. Accordingly, the formula (15) is expected to represent the wave component of (1). Comparing (15) with Eq.27b in [6], all results obtained in [2] on the wave component remain valid. The additional term $\mathcal{E} = \exp(-\mu B\varphi')$ in the amplitude function does not introduce any modification to the wave form. As an exponentially decreasing (damping) term, it represents only the dissipative effect.

Steady ship waves

We consider the special case of steady flow for which $f = 0 = \tau$ so that the real dispersion function (3) becomes $D = k(k\cos^2\theta - 1 - \sigma^2 k^2)$ and the imaginary part (4) becomes $B = -4k^3\cos\theta$ in polar coordinates (k, θ). Only two dispersion curves $(D = 0)$ exist and are symmetrical with respect to both axes $\alpha = 0$ and $\beta = 0$. In the quadrant $\alpha \geq 0$ and $\beta \geq 0$, the dispersion curve is defined explicitly

$$k(\theta) = \begin{cases} k_g(\theta) = 2/\big(\cos^2\theta + \sqrt{\cos^4\theta - 4\sigma^2}\,\big) & k \leq k_\sigma \\ k_T(\theta) = \big(\cos^2\theta + \sqrt{\cos^4\theta - 4\sigma^2}\,\big)/(2\sigma^2) & k \geq k_\sigma \end{cases} \tag{16}$$

with $k_\sigma = 1/\sigma$. The curve described by (16) is a closed one limited in the region

$$0 \leq \theta \leq \theta_\sigma \qquad \text{with} \quad \theta_\sigma = \arctan\left[\sqrt{(1-2\sigma)/(2\sigma)}\,\right] \tag{17}$$

At $\theta = \theta_\sigma$ we have $k = k_\sigma$. At $\theta = 0$, we define

$$k_g^0 = 2/\big(1+\sqrt{1-4\sigma^2}\,\big) \quad \text{and} \quad k_T^0 = \big(1+\sqrt{1-4\sigma^2}\,\big)/(2\sigma^2) \tag{18}$$

so that the dispersion curve intersects the α-axis at $\alpha = k_g^0$ and $\alpha = k_T^0$.

The dispersion curves given by (16) are depicted on the *left* part of Fig.1 at a Froude number $F = 0.1$ (using $L = 1$ m) for $\sigma = 0$ when the surface tension is ignored and $\sigma = 0.275$ when the surface tension is included. The dispersion curve without the surface tension $(\sigma = 0)$ represented by the dashed line is given by $k = 1/\cos^2\theta$ and corresponds to the case usually called Neumann-Kelvin ship waves. It is an open curve as $k \to \infty$ when $\theta \to \pi/2$. The dispersion curve with the surface tension

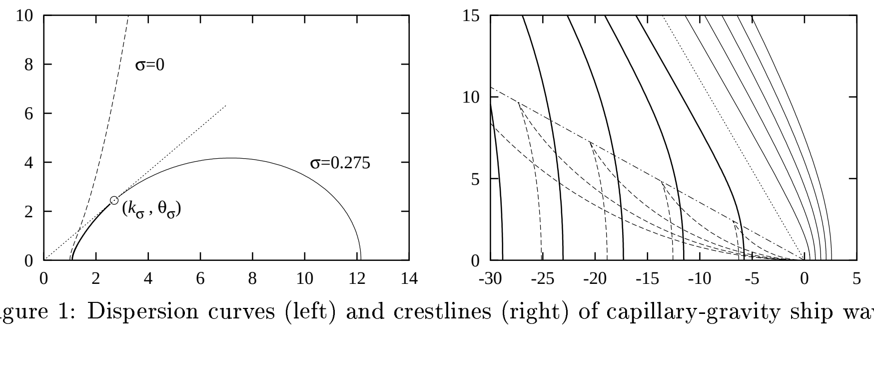

Figure 1: Dispersion curves (left) and crestlines (right) of capillary-gravity ship waves

$(\sigma \neq 0)$ is a *closed* one with a maximum wavenumber k_T^0 defined by (18). The point $(k_\sigma, \theta_\sigma)$ divides the dispersion curve into two portions : one ($k_g < k_\sigma$, thick solid line) along which the effect of gravity is dominant and another ($k_T > k_\sigma$, thin solid line) along which the capillarity is dominant.

On the *right* part of Fig.1, we depict the crestlines following the formulation (Eq.15 in [2]) for $n = (1, 2, \cdots, 5)$ associated with the dispersion curves plotted on the left part of the figure. The Neumann-Kelvin ship waves represented by dashed lines composed of transverse and divergent waves are present only in the downstream and limited by a cusp line (dot-dashed line). The ship waves including the surface tension are present in both upstream and downstream. The upstream crestlines associated with the part of dispersion curve at $k_T > k_\sigma$ are capillary waves and plotted by thin solid lines. The wavelength of upstream capillary waves is of order $2\pi F^2/k_T^0$.

The downstream crestlines (thick solid lines) associated with the part of dispersion curve at $k_g < k_\sigma$ are gravity-dominant waves. Comparing to the pure-gravity waves (dashed lines), the transverse waves keep the same profile with a slight shorter wavelength $2\pi F^2/k_g^0$ instead of $2\pi F^2$. The most striking feature concerns the divergent waves which disappear completely at this value of σ (in fact for $\sigma > \sigma_0$ given in the following) due to the effect of surface tension. In their place, the transverse waves are extended smoothly outward to a region limited by the ray (dotted line) forming an angle γ_σ with the negative-x axis defined in

$$\gamma = \arctan[y/(-x)] \leq \gamma_\sigma = \pi/2 - \theta_\sigma \tag{19}$$

The crestlines for $n = (1, 2, \cdots, 5)$ are depicted on Fig.2 for $\sigma = 0.02$ (left part). Only those of downstream waves are drawn for the sake of clarity. The transverse waves are represented by thick solid lines and the divergent waves by thin solid lines, while the rest of capillary-gravity waves by dashed lines limited by the dotted ray ($\gamma = \gamma_\sigma$). $\gamma_\sigma = 0$ at $\sigma = 0$ means that no capillary waves exist since the effect of surface tension is ignored. At $\sigma = \sigma_m = 1/2$, the dispersion curve reduces to a point $(2, 0)$ and $\gamma_\sigma = \pi/2$ which means that all steady waves disappear (no wavy deformation of the free surface) since ship's speed is less than the minimum velocity of capillary-gravity waves so that waves propagating at ship's speed cannot be generated.

There are two other important rays, more evident on the right part of Fig.2 on which only crestlines of divergent waves are kept. One represented by the thin

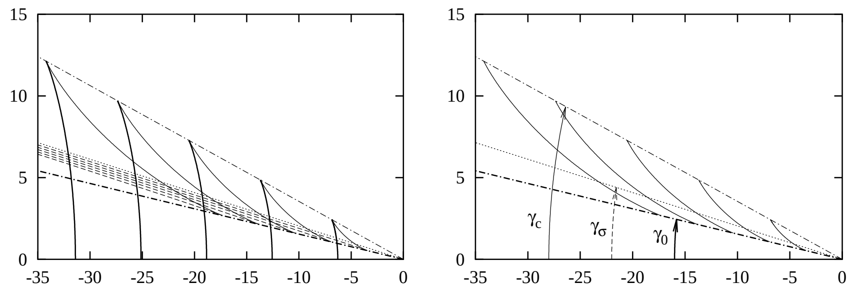

Figure 2: Crestlines of capillary-gravity ship waves (left) and definition of rays (right)

dot-dashed line is close to the cusp line of Kelvin ship waves, and another by thick dot-dashed line. We denote the two rays respectively by $\gamma = \gamma_c$ and $\gamma = \gamma_0$ the angles forming with the negative-x axis. Same as γ_σ, the ray-angles γ_c and γ_0 are function of the parameter σ. Following Eq.13 in [2], the value of γ_c is associated with the normal direction at the first point of inflection along the dispersion curve, which is quite close to that for the Neumann-Kelvin ship waves. There exists a second inflection point along the dispersion curve of capillary-gravity ship waves at low values of σ. The value of γ_0 is given by the normal direction at this second point of inflection. The ray-angle γ_c becomes the cusp angle $\gamma_c^0 = \gamma_c(\sigma = 0) \approx 19°28'$ of pure-gravity ship waves when $\sigma \to 0$ while γ_0 tends to zero. It is shown that the divergent waves can be found only in the region $(\gamma_0 \leq \gamma \leq \gamma_c)$ where transverse waves appear as well. In the region near the ship's track $(0 \leq \gamma \leq \gamma_0)$, only transverse waves are present. Since γ_0 increases significantly with increasing σ (corresponding to the decrease of forward speed), the region $(\gamma_0 < \gamma < \gamma_c)$ where divergent waves appear is more and more reduced. At $\sigma = \sigma_0 \approx 0.133$ (corresponding to $U = U_0 \approx 0.450$ m/s), there does not exist any divergent wave.

The damping term $\mathcal{E} = \exp(-\mu B \varphi')$ due to fluid viscosity can be evaluated by using (11) for φ' and $B = -4k^3 \cos\theta$. In particular, the gravity-dominant transverse waves in the downstream is dissipated at a rate proportional to

$$\mathcal{E}_d \approx \exp\left[\mu 4 (k_g^0)^2 x\right] \quad \text{with} \quad k_g^0 \approx 1 \quad \text{for} \quad x < 0 \tag{20}$$

while the amplitude of capillary-dominant waves in the upstream is damped at a rate proportional to

$$\mathcal{E}_u \approx \exp\left[-\mu 4 (k_T^0)^2 x\right] \quad \text{with} \quad k_T^0 \approx 1/\sigma^2 \quad \text{for} \quad x > 0 \tag{21}$$

To reduce by a factor of e through the viscosity dissipation, the distances to travel from the singularity are $|x_d| \approx 1/(4\mu)$ for the gravity waves and $x_u \approx \sigma^4/(4\mu)$ for capillary waves, which is σ^{-4} times shorter than that for gravity waves!

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