CONVENTIONAL AND LINEAR STATISTICAL MOMENTS APPLIED IN EXTREME VALUE ANALYSIS OF NON-GAUSSIAN RESPONSE OF JACK-UPS

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ABSTRACT

This work investigates numerically two different methods of moments applied to Hermite derived probability distribution model and variations of Weibull distribution fitted to the short-term time series peaks sample of stochastic response parameters of a simplified jack-up platform model which represents a source of high non-Gaussian responses. The main focus of the work is to compare the results of short-term extreme response statistics obtained by the so-called linear method of moments (L-moments) and the conventional method of moments using either Hermite or Weibull models as the peaks distribution model. A simplified mass-spring system representing a three-legged jack-up platform is initially employed in order to observe directly impacts of the linear method of moments (L-moments) in extreme analysis results. Afterwards, the stochastic response of the three-legged jack-up platform is analyzed by means of 3-D finite element model. Bias and statistical uncertainty in the estimated extreme statistics parameters are computed considering as the “theoretical” estimates those evaluated by fitting a Gumbel to a sample of episodic extreme values obtained from distinct short-term realizations (or simulations). Results show that the variability of the extreme results, as a function of the simulation length, determined by the linear method of moments (L-moments) is smaller than their corresponding ones derived from the conventional method of moments and the biases are more or less the same.

INTRODUCTION

The non-Gaussian or nonlinear response time-series are often found in the stochastic analysis of many types of offshore structures [1] submitted to environmental loadings; this source of nonlinearity may be caused by many factors as soil-structure interaction, free surface wave effects, hydrodynamic loading nonlinear effects, etc. Specifically in this work, the main source of nonlinearity comes from the square fluid-structure velocity term in the drag component of Morison’s equation and from the free surface wave effects. As, in general, no analytical solutions are available for the estimation of extreme statistics of non-Gaussian responses, a large number of practical numerical approaches are nowadays available in literature for this purpose, such as Weibull-based methods [2], Hermite polynomials-based approach [3], ACER method [4]. Weibull and Hermite models employ sample statistical moments of the sample peaks and of the whole time-series, respectively, which are prone to statistical uncertainties due the limited size of the sample.

Recently, Hosking [5] proposed the application of the so-called linear method of moments or L-moment in the estimation of distribution parameters as an alternative to the conventional method of moments (mean, variance, coefficient of skewness and kurtosis). Due to the low order estimates, it is expected that L-moments present less statistical uncertainties and, consequently, any extreme estimate based on them could present low variability when compared with estimates based on conventional statistical moments.

The purpose of this work is to investigate numerically the use of the L-moments in extreme response analysis of a Jack-up
platform model and compare the results obtained with previous results presented in Nascimento et al. [2] where conventional method of moments has been applied on extreme value estimations using Hermite and Weibull approaches. To accomplish with this objective the most probable extreme value (MPEV) of the Jack-up surge is analyzed. A reference “theoretical” value for comparisons is obtained by fitting a Gumbel distribution fitted to a sample of episodic extremes taken from various independent realizations (non-linear time dynamic simulations).

DERIVATION OF DISTRIBUTION PARAMETERS BY L-MOMENTS

One approach to define a theoretical probability model for a given data sample of a random variable is by means of the method of moments. The classical method of moments consists in computing the parameters of the probability model in order to match the sample n-order moments such as mean, variance, coefficients of skewness and kurtosis [6]. In extreme statistics applied to non-Gaussian time series the analytical model based on Hermite-polynomials [3] also employs the four first sample coefficients of skewness and kurtosis [6]. In extreme statistics to match the sample n-order moments such as mean, variance, in computing the parameters of the probability model in order to accomplish with this objective the most probable extreme value estimations using Hermite and Weibull approaches. To method of moments has been applied on extreme value results presented in Nascimento et al. [2] where conventional platform model and compare the results obtained with previous estimates presented in the study [2].

L-moments are expectations of certain linear combinations of order statistics [5]. Let Y be a real-valued random variable with cumulative distribution function F(y) and quantile function y(F), and let \( Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n} \) be the order statistics of a random sample of size n drawn from the distribution of Y. L-moments of Y are defined to be the binomial quantities

\[
\lambda_r = n^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E[Y_{(r-k):r}] \quad r = 1, 2, \ldots
\]

The expectation of an order statistic may be written as

\[
E[Y_{(r):r}] = \frac{r!}{(j-1)!(r-j)!} \int y \cdot [F(y)]^{r-1} [1 - F(y)]^{j-1} dF(y)
\]

(2)

The theory implies that \( \lambda_r \) is a linear function of the expected order statistics \( E[Y_{(r-k):r}] \). In particular, the first four L-moments are

\[
\lambda_1 = E[Y] \\
\lambda_2 = \frac{1}{2} E[Y^2] - E[Y]^2 \\
\lambda_3 = \frac{1}{3} E[Y^3] - 2 E[Y^2] + 3 E[Y] \\
\lambda_4 = \frac{1}{4} E[Y^4] - 3 E[Y^3] + 6 E[Y^2] - 4 E[Y]
\]

(3)

From the distribution theory of the order statistics \( Y_{(r-k):r} \), \( \lambda_r \) can be rewritten in terms of either F(y) and y(F)

\[
\lambda_r = \int_{F_{(r)}}^{F_{(n)}} y(F) w_r(F) dF = \int_{y_{(r)}}^{y_{(n)}} y w_r[F(y)] f(y) dy
\]

(4)

The weights \( w_r[.] \) are polynomial functions of F in the following form

\[
w_r[F] = \sum_{k=0}^{r} p_{r,k} F^k
\]

(5)

where

\[
p_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}
\]

(6)

The estimation of the L-moments from a sample of size n from the random variable Y is discussed in [7]. Commonly used estimates are

\[
\lambda_1 = \frac{1}{n} \sum_{i=1}^{n} y_{i:n} = \bar{Y}
\]

\[
\lambda_2 = \frac{1}{n} \sum_{i=1}^{n} (2P_{i:n} - 1)y_{i:n}
\]

(7)

\[
\lambda_3 = \frac{1}{n} \sum_{i=1}^{n} (6P_{i:n}^2 - 6P_{i:n} - 1)y_{i:n}
\]

\[
\lambda_4 = \frac{1}{n} \sum_{i=1}^{n} (20P_{i:n}^2 - 30P_{i:n} + 12P_{i:n} - 1)y_{i:n}
\]

where \( P_{i:n} \) is a distribution-free estimator of the value of the cumulative distribution function of the random variable Y at \( y_{i:n} \); that is, \( P_{i:n} \) is an estimate of \( v(y_{i:n}) \). In this study \( P_{i:n} \) was taken to be

\[
P_{i:n} = \frac{i - 0.4}{n + 0.2}
\]

(8)

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\( \lambda_1 \) and \( \lambda_2 \) are measures of the central trend and dispersion. Non-zero \( \lambda_3, \lambda_4, \ldots \) reflect deviations of the distribution of \( Y \) from a uniform density, where \( \lambda_3 \) and \( \lambda_4 \) reflect asymmetric and symmetric deviations, respectively. Similarly to the classical statistical moments, The “linear skewness” (L-skewness) and “linear kurtosis” (L-kurtosis) are given, respectively, by

\[
\tau_3 = \frac{\lambda_3}{\lambda_2} \quad \tau_4 = \frac{\lambda_4}{\lambda_2} \quad (9)
\]

**The L-moments for Gaussian Variables**

Consider the special case of a standard normal variable, denoted as \( U \), with cumulative distribution function, \( F(u) = \Phi(u) \) and probability density function \( \phi(u) = \left(\sqrt{2\pi}\right)^{-1} \exp\left(-0.5u^2\right) \). From Eq.(4), its L-moments are of the form

\[
\lambda_r = \frac{\int u \cdot w_r \cdot \phi(u) \cdot u \cdot w_r \cdot \phi(u) \cdot du}{\int \phi(u) \cdot u \cdot w_r \cdot \phi(u) \cdot du} \quad (10)
\]

To simplify this result, the cumulative distribution function \( \Phi(u) \) can be expressed as 0.5\[1 + \text{erf}\left(u/\sqrt{2}\right)\], in terms of error function \( \text{erf}(\cdot) \). By substituting this equation into Eq.(5), the weight functions in the Gaussian case become

\[
\begin{align*}
w_1[\Phi(u)] &= 1 \\
w_2[\Phi(u)] &= \text{erf}\left(\frac{u}{\sqrt{2}}\right) \\
w_3[\Phi(u)] &= 1.5\text{erf}^2\left(\frac{u}{\sqrt{2}}\right) - 0.5 \\
w_4[\Phi(u)] &= 2.5\text{erf}^2\left(\frac{u}{\sqrt{2}}\right) - 1.5\text{erf}\left(\frac{u}{\sqrt{2}}\right)
\end{align*}
\]

Since \( w_1[\Phi(u)] \) and \( w_2[\Phi(u)] \) are even functions of \( u \); \( u, w_1[\Phi(u)] \) and \( u, w_2[\Phi(u)] \) are odd, then \( \lambda_1 = \lambda_3 = 0 \). The non-zero L-moments, \( \lambda_2 \) and \( \lambda_4 \), are then given by

\[
\begin{align*}
\lambda_2[U] &= \frac{1}{\sqrt{\pi}} = 0.56419; \lambda_4[U] = 0.06917 \\
\lambda_2[U] &= \frac{1}{\sqrt{\pi}} = 0.56419; \lambda_4[U] = 0.06917
\end{align*}
\]

The corresponding L-skewness and L-kurtosis are

\[
\tau_3 = 0; \quad \tau_4,\text{Gauss} = \frac{0.06917}{0.56419} = 0.1226 \quad (13)
\]

**The L-moments for Weibull-distributed variables**

One statistical model largely used in offshore engineering to model the peaks (see Figure 1) of a sampled time series is the Weibull distribution. The cumulative distribution function of the Weibull distribution is given by

\[
F_Y(y) = 1 - \exp\left(-\left(\frac{y - \xi}{\alpha}\right)^\kappa\right) \quad (14)
\]

where \( \xi, \alpha \) and \( \kappa \) are, respectively, the location, scale and shape distribution parameters. Using the traditional method of moments these parameters can be directly related to the mean, standard deviation and skewness coefficient of sample data [8]. On the other hand, the relationship between the L-moments and these distributions parameters is given in Greenwood [9] and are expressed by

\[
\begin{align*}
\xi &= \lambda_1 + \frac{5\lambda_2}{\tau_4 - 5\tau_3 - 1} \\
\alpha &= \log\left(\frac{10}{\log(\tau_4 - 5\tau_3 - 1)}\right) \\
\kappa &= \frac{\log(2)}{\log(\tau_4 - 5\tau_3 - 1)}
\end{align*}
\]

\( \tau_4,\text{Gauss} = 0.56419 \)

The estimation of the extreme short-term statistics of a response parameter whose peaks sample has been modeled by a Weibull model is made through Statistics of Extreme Asymptotic Theory [8]. In this case it has been shown that the most probable short-term extreme peak value of a sampled time series of a response parameter is given by

\[
MPEV = \xi + \alpha [\ln(\nu^+T)]^{\frac{1}{\kappa}}
\]

where \( \nu^+ \) is the average rate of peaks of the response parameter and \( T \) is short-term period considered (usually, 3-h).

**The L-moments for Hermite-based model**

The Hermite polynomials-based model, developed by Winterstein [10] is largely employed in the prediction of extremes and fatigue associated to non-Gaussian sampled time series. The main idea is to transform through a memoryless
nonlinear mapping, based on Hermite polynomials, a non-Gaussian time series into a Gaussian time series. Considering a non-Gaussian sampled time series $Y$, its normalized counterpart $y_i$ is defined as:

$$y_i = \frac{Y_i - m_Y}{s_Y}$$  \hspace{1cm} (19)$$

where $m_Y$ and $s_Y$ are, respectively, the sample mean and standard deviation of $Y$. The statistical equivalence between $y(t)$ and a standard Gaussian time series $u(t)$ is expressed as

$$y(t) = \sum_{n=1}^{N} c_n H_{n-1}(u(t))$$  \hspace{1cm} (20)$$

where $H_{n-1}(.)$ is the n-order Hermite polynomial, $c_n$ is its associated coefficient and $N$ stands for the number of polynomials considered in the approximation. Considering four (4) terms in the approximation, $y$ can be written as

$$b = \frac{c_3}{(1 - 3c_4)}; c = \frac{c_4}{(1 - 3c_4)}$$

$$y = m_Y + s_Y \kappa [u + b(u^2 - 1) + cu^3]$$  \hspace{1cm} (21)$$

In which $U$ is standard normal, and $m_Y$ and $s_Y^2$ are the mean and variance of the sampled time series $Y$ and

$$\kappa = \frac{1}{\sqrt{1 + 2b^2 + 6c + 15c^2}}$$  \hspace{1cm} (22)$$

In order to compute the coefficients $c_3$ and $c_4$, the skewness and kurtosis coefficients of the sampled time series $Y$, can be employed [10]. In this work the coefficients $c_3$ and $c_4$ are fit numerically so skewness and kurtosis coefficients are preserved. Extreme statistics estimates for $y(t)$ are obtained by standard analytical solutions for the extremes of $u(t)$ and then using Eq. (21).

As described above, the Hermite-based model uses the conventional statistical moments of the sample time series $Y(t)$. However, as shown by Winterstein and McKenzie [11] for L-Hermite coefficients, feasible results for this specific application were not obtained and among the possibilities, the methodology applied in this work proposes that the coefficients $c_3$ and $c_4$ be computed through standard numerical procedures preserving original skewness coefficient and applying L-kurtosis instead of kurtosis. As the main source of nonlinearity comes from Morison’s equation, the parameter $\gamma_2$ has been computed by an approximation proposed by Najafian [7]. As can be seen in [7], eq.(25) showed to be much more efficient than conventional method of moments for high-kurtosis distributions

$$\gamma_2 = -365.906 r_4^3 + 211.333 r_4^2 + 2.385 r_4 + 0.190$$  \hspace{1cm} (25)$$

where $\gamma_2$ is the kurtosis coefficient of the response sampled time history.

**NUMERICAL APPLICATIONS: JACK-UP MODELS**

In this section the most probable extreme short-term responses of two Jack-up models are evaluated by means of a Weibull approach and by the Hermite-based model using both conventional and linear statistical moments of the simulated response time-series. The first example is a single degree of freedom model, described by Jensen and Capul [12], idealized to represent the sway motion of a Jack-up. The other one is a 3-DOF simplified model of a Jack-up operating in a water depth of 92.0m. The Weibull model employed consists in fitting a 3-parameter Weibull distribution to a subset of the sampled time series peaks which are above a given threshold. In this work this threshold is taken as the value corresponding 75% fractile of the empirical cumulative distribution of the sample containing all time series peaks. Hereafter, this approach is identified simply as Weibull-3P-ToF. As shown in Nascimento et al. [2], among various Weibull-based approaches this is one with better performance in extreme predictions for the response offshore structures whose main source of nonlinearity comes from Morison’s hydrodynamic loading.

**Single Degree of Freedom Model of a Jack-up**

The first Jack-up model analyzed corresponds to a very simplified single degree of freedom (SDOF) system presented by Jensen and Capul [12], representing the lateral behavior of a 120m high three-legged jack-up platform located in a water depth of 92.0m. The model is described by the following differential equilibrium equation

$$m \ddot{x} + c \dot{x} + kx = F(t)$$  \hspace{1cm} (26)$$

where $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are, respectively, the jack-up lateral motion, velocity and acceleration, $m$ is the total model mass (18995 kN.s/m), $c$ is the model damping (1,193x10^{-3} kN.s/m), $k$ is the lateral stiffness (4,682x10^5 kN/m) and $F(t)$ is an idealized Morison-type force acting on the structure. Using this data one can see that the natural frequency $\omega_0$ of the model is
1.57 rad/s and the damping ratio is 0.02. The short-term environmental condition considered is represented by a sea state having a significant wave height (Hs) of 6.3m and zero-crossing period (Tz) of 8.4s. The sea state spectrum is modeled by the modified Pierson-Moskovitz spectrum. The short-term duration is considered to be 3-h (10800s). F(t) has been generated artificially in order cover inertia and drag dominated cases of the hydrodynamic loading through a K factor define as

$$K = \frac{k_d s_d}{k_m s_u}$$  \hspace{1cm} (27)

where $s_d$ and $s_u$ are the standard deviations of fluid velocity and acceleration, respectively; $k_d$ and $k_m$ are the drag and inertia parameters. It is observed that in the case of the inertia dominated loading $K \to 0$ and $K \to \infty$ when it is drag dominated. In this work three different values of K have been simulated i.e., K = 3.00, K = 1.00 and K = 0.25.

Initially, for each K value, 100 distinct 10800-s long sample time series of lateral motion x(t) have been simulated through numerical integration of Eq.(26) by the Euler method [13]. The largest episodical values of each generated lateral motion time-series has been taken to form a sample of 3-h extreme values. Using the method of moments [14], a Gumbel extreme value distribution has been fitted to the data and the extreme values. Using the method of moments [14], a Gumbel extreme value distribution has been fitted to the data and the most probable 3-h extreme value (MPEV) identified. This value has been taken as the reference value for the analysis of the results obtained by both Hermite and Weibull-3P-PoT methods.

Hermite and Weibull-3P-PoT estimates of the 3-h most probable extreme surge motion have also been estimated using traditional and linear statistical moments different simulation lengths. Tables 1-3 present the mean value of the 100 3-h MPEVs estimated by these approaches divided by the 3-h MPEV value obtained from the extreme sample. These tables also include (in brackets) the 95% confidence of variance (CoV) of the estimators.

**Table 1.** Normalized MPEVs estimated for SDOF Jack-up Model for K = 3.00.

<table>
<thead>
<tr>
<th>Simulation Length (s)</th>
<th>Conventional moments</th>
<th>L-moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hermite</td>
<td>Weibull 3P-PoT</td>
</tr>
<tr>
<td>1200</td>
<td>1.09</td>
<td>(±0.339)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(±0.272)</td>
</tr>
<tr>
<td>2400</td>
<td>1.11</td>
<td>(±0.172)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(±0.133)</td>
</tr>
<tr>
<td>4800</td>
<td>1.12</td>
<td>(±0.112)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(±0.076)</td>
</tr>
<tr>
<td>7200</td>
<td>1.12</td>
<td>(±0.067)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figures 2-4 present graphically the bias and 95% confidence interval for the results obtained with Weibull-3P-PoT model for K = 3.00, K = 1.00 and K = 0.25, respectively. It should be noticed that the ideal method should have a bias around 1.0 and a confidence interval very small.
As already shown [2], the Hermite model predicts biased results mainly for drag-dominated situations. However, in all cases analyzed for this example it is observed that the statistical uncertainty is a little bit smaller for predictions based on L-moments.

3-D Simplified Finite Element Model of a Jack-Up

The extreme estimates using L-moment method described before are now investigated in the analysis of the surge motion and base shear force of a 3-D Jack-up model operating in a water depth of 92.0m. Due to computational effort, quantity of generated data and computer code restrictions [15] a simplified jack-up computer finite element-based model (shown in Fig. 5) has been used. The 3-D finite-element model is made-up of 130 beam elements and 109 nodes, representing the platform legs and hull [16]. Also due to these computer analysis limitations, the short-term period has been considered to be equal to 6000s.
method of moments. Figs. 6-7 illustrate graphically these normalized results for Weibull-3P-PoT. It is observed that the Weibull-based predictions lead to less biased results. However, the most important aspect to be pointed out is that the reduction in the statistical uncertainty of the predicted extremes using L-moments in connection with Weibull-3P-PoT approach is not expressive, i.e., this statistical uncertainty does not differ too much when using conventional or linear statistical moments in the extreme predictions. The reduction is a little bit larger in the case of the Hermite-based model but on the other hand this approach presents larger biases.

### Table 4. Normalized lateral displacement 6000s MPEVs estimated for the 3-D Jack-up model.

<table>
<thead>
<tr>
<th>Simulation Length (s)</th>
<th>Conventional moments</th>
<th>L-moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hermite</td>
<td>Weibull 3P-PoT</td>
</tr>
<tr>
<td>1200</td>
<td>0.95 (±0.154)</td>
<td>0.83 (±0.151)</td>
</tr>
<tr>
<td>2400</td>
<td>1.22 (±0.259)</td>
<td>1.05 (±0.239)</td>
</tr>
<tr>
<td>4800</td>
<td>1.21 (±0.151)</td>
<td>1.03 (±0.143)</td>
</tr>
<tr>
<td>6000</td>
<td>1.17 (±0.136)</td>
<td>1.01 (±0.133)</td>
</tr>
</tbody>
</table>

### Table 5. Normalized base shear force 6000s MPEVs estimated for the Jack-up Model.

<table>
<thead>
<tr>
<th>Simulation Length (s)</th>
<th>Conventional moments</th>
<th>L-moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hermite</td>
<td>Weibull 3P-PoT</td>
</tr>
<tr>
<td>1200</td>
<td>0.92 (±0.159)</td>
<td>0.81 (±0.145)</td>
</tr>
<tr>
<td>2400</td>
<td>1.19 (±0.267)</td>
<td>1.05 (±0.261)</td>
</tr>
<tr>
<td>4800</td>
<td>1.19 (±0.151)</td>
<td>1.03 (±0.159)</td>
</tr>
<tr>
<td>6000</td>
<td>1.15 (±0.136)</td>
<td>1.00 (±0.145)</td>
</tr>
</tbody>
</table>

### CONCLUSION

In this study the main focus was to investigate numerically the application of the so-called L-moments method [5] applied to Weibull and Hermite distribution models in the extreme value prediction of non-Gaussian responses of Jack-up platforms. The best Weibull-based approach for this type of structure, discussed in a previous work [2], the Weibull-PoT, has been used in this study.

Based on the analyses of a very simplified single degree of freedom (SOF) model of a Jack-up and another 3-D finite element-based model of another Jack-up platform the main conclusion that can be drawn from the present study is that application of the L-moments method leads to lower variability in MPEV results when compared those predicted using conventional statistical moments; however, this reduction is not so expressive.
REFERENCES


